Heterotic M(atrix) theory at generic points in Narain moduli space

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Abstract

Type II compactifications with varying string coupling can be described elegantly in F-theory/M-theory as compactifications on U - manifolds. Using a similar approach to describe Super Yang-Mills with a varying coupling constant, we argue that at generic points in Narain moduli space, the $E_8 \times E_8$ Heterotic string compactified on T^2 is described in M(atrix) theory by $\mathcal{N} = 4$ SYM in 3+1 dimensions with base $S^1 \times \mathbb{CP}^1$ and a holomorphically varying coupling constant. The $\mathbb{CP}^1$ is best described as the base of an elliptic K3 whose fibre is the complexified coupling constant of the Super Yang-Mills theory leading to manifest U-duality. We also consider the cases of the Heterotic string on S^1 and T^3. The twisted sector seems to (almost) naturally appear at precisely those points where enhancement of gauge symmetry is expected and need not be postulated. A unifying picture emerges in which the U-manifolds which describe type II orientifolds (dual to the Heterotic string) as M- or F- theory compactifications play a crucial role in Heterotic M(atrix) theory compactifications.

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1 Introduction

Our improved understanding of duality symmetries, both perturbative and non-perturbative, suggests that the five string theories as well as eleven dimensional supergravity are limits of a single theory, an eleven dimensional theory called M-theory. Some insight into this theory has been provided by looking at strong coupling limits of various string theories[1]. A new tool in understanding M-theory was provided by Banks, Fischler, Shenker and Susskind[2] who introduced a M(atrix) theory which was obtained by considering the strong coupling limit of N D0-branes in IIA string theory. The quantum mechanics of this M(atrix) theory[3] was proposed as a candidate for M-theory in the Infinite Momentum Frame (IMF) with 'N', the number of D0-branes being identified with the eleven dimensional momentum. This theory has passed a number of tests. We refer the reader to a recent review[4] for a description of the details of those results.

Further compactifications of M(atrix) theory on tori T^d have been useful in understanding aspects of this theory. For compactifications where $d \leq 3$, this leads to Supersymmetric Yang-Mills (SYM) in $d+1$ dimensions with base manifold given by the dual torus $\widetilde{T}^d$.[5, 6, 7] More importantly, U-duality[2] is visible in M(atrix) theory with excited states sitting in representations of the appropriate U-duality group. For $d > 3$, the SYM prescription breaks down. This breakdown can be seen in two ways: The full U-duality group is not realised and these are non-renormalisable quantum field theories. Both problems suggest the need for additional degrees of freedom to make the SYM prescription complete. This has led to the construction of new theories which exhibit manifest U-duality[9, 10]. One important consequence is that the base manifold of M(atrix) theory is related to the target spacetime only in special limits. For example, even a simple topological quantity like the dimension of the (compact part of) target spacetime and that of the base manifold may not be the same.

Another class of interesting compactifications of M(atrix) theory are those which involve lesser supersymmetry. The simplest class of these models is Heterotic M(atrix) theory and its compactifications[11, 12]. Other examples are compactification of M-theory on K3, IIB on K3, F-theory compactified

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[2]This is a conjectured discrete symmetry which includes all perturbative and non-perturbative symmetries of string theory[8].

on elliptic K3 as well as non-compact examples such as ALE spaces and non-compact orbifolds[13]. These theories all have eight unbroken supercharges and hence supersymmetry imposes fewer constraints here. Thus these models are still not as well understood as the case with sixteen unbroken supercharges except at special points in their moduli spaces. However, recently there has been progress for the case of Heterotic M(atrix) theory on S^1[14]. It has also been noticed that some of the known non-perturbative dualities have interesting realisations in M(atrix) theory. For example, M-theory on K3 and Heterotic string on T^3 are different limits of the same M(atrix) theory[15, 16, 17].

Even though several of our considerations will be rather general, the main focus of this paper will be the case of Heterotic M(atrix) on T^2. We will also have some remarks for the case of compactification on S^1 which has been recently studied by Kabat and Rey[14] and for the case of T^3 compactification. The moduli space of Heterotic string theory compactified on T^d is given by the Narain moduli space[18]

$$\Gamma_d \backslash \frac{SO(16+d, d)}{SO(16+d) \times SO(d)} \times \mathbb{R}^+ \quad , \tag{1}$$

where $\mathbb{R}^+$ corresponds to the heterotic dilaton and $\Gamma_d = SO(16+d, d; \mathbb{Z})$ is the U-duality group for these theories(for $d \leq 3$). For $d = 1$, there are 18 real moduli and for $d = 2$, there are 18 complex moduli (excluding the dilaton). Of these, 16 complex (real) moduli correspond to the Wilson lines of $E_8 \times E_8$ for d=2 (d=1). A complete description of the Heterotic M(atrix) compactified on T^d should include a manifest realisation of this symmetry. It is also clear that such a description will include generic points in Narain moduli space.

The working prescription for describing the Heterotic string on T^d in M(atrix) theory is to consider the orbifold U(N) SYM in $d+2$ dimensions with base $\widetilde{S}^1 \times (\widetilde{T}^d/\mathbb{Z}_2)$[11, 12]. Anomaly cancellation is accomplished by including a twisted sector given by 32 fermions distributed equally among the 2^d fixed points (circles to be precise) of the orbifold $\mathbb{Z}_2$. This choice corresponds to a point in Narain moduli space with enhanced gauge symmetry $[SO(32/r)]^r$, where $r = 2^d$. Since the prescription assumes the U(N) SYM description for M(atrix) theory on T^{d+1}, we will restrict to the cases $d \leq 2$ though we will

have some remarks for $d = 3$. [3]

The problem of describing M(atrix) theory at more generic points in Narain moduli space can be described in two ways: The first is to view it as a clear description of Wilson lines in Heterotic M(atrix) theory. The second is to view it as providing a manifest U-dual description of Heterotic M(atrix) theory. These are complementary views and hence understanding one should shed light on the other. This paper pursues the second perspective. An important observation (see Sec. 4 for details) to make here is that for both $d = 1, 2$ (as well as for other values of $d < 5$), the Heterotic string at the point in moduli space where the enhanced gauge symmetry is $[SO(32/r)]^r$, the theory is dual to orientifolds of type II string theory (IIA for odd d and IIB for even d)[19]. Further, this duality maps Wilson lines to the positions of certain D-branes which have to be included into the orientifolding for cancelling RR charges located at the fixed points of the orientifold group. In these cases, generic points in the moduli space correspond to moving the D-branes away from the orientifold fixed point. However, this involves considering moduli which depend on the compactified coordinates and includes regions where one is at strong coupling[20, 21].

How does one view varying spacetime moduli in M(atrix) theory. Generically, one expects that these moduli will enter as parameters in the base manifold. On D-brane probes, one sees however that varying moduli (such as the dilaton) lead to a varying coupling constant in the worldvolume theory. *We shall make the ansatz that varying moduli in spacetime should correspond to introducing varying coupling constants in M(atrix) theory.* This was first proposed by Kabat and Rey in the context of Heterotic M(atrix) theory on S^1[14]. Thus our problem can be posed as the need to understand the situation of SYM with a varying coupling constant such that U-duality is manifest with the appropriate amount of supersymmetry being broken.

This might seem to be a rather difficult problem to solve. We solve this by a method used in F-theory[22] by considering a $d + s$ dimensional fibred manifold $\mathcal{Y}$ (we will refer to this manifold as the U-manifold following [23]) with base $\mathcal{B}$ of dimension d and fibre is identified with the coupling constant of the SYM theory. Now consider compactifying the M(atrix) theory which

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[3]The cases of $d = 3, 4$ can in principle be included by using the orbifolding procedure on the theories proposed for M(atrix) theory on T^4 and T^5[9, 10].This will however require a more detailed understanding of these new theories.

describes M-theory on T^r (for some $r > d + 1$) and compactify it on a base $S^1 \times \mathcal{Y}$. In the limit of large base and non-singular fibres, this can be viewed as SYM with base $S^1 \times \mathcal{B}$ and a coupling constant varying with respect to the coordinates on $\mathcal{B}$ as dictated by the geometry of $\mathcal{Y}$. An important constraint on $\mathcal{Y}$ is that there must be a special point in its moduli space where $\mathcal{B}$ can be identified with $T^d/\mathbb{Z}_2$ where the fibre is constant. At this point, we recover the standard prescription of SYM on $S^1 \times (T^d/\mathbb{Z}_2)$[11, 12]. Other conditions on $\mathcal{Y}$ are that it break half the supersymmetry and have a moduli space which can be identified with the Narain moduli space. Though the above is probably true in general we will discuss the candidates for $d = 1, 2, 3$ in this paper as concrete examples. In these cases, we will see that the results seem consistent with existing proposals for Heterotic M(atrix) theory based on known string dualities[15, 16, 17]. Further, we will see that the twisted sector emerges without being postulated at precisely the points where enhanced gauge symmetry occurs (though we do not understand the $d = 1$ case very well).

The plan of this paper is as follows. In Sec. 2, we review some of the relevant aspects of Heterotic M(atrix) theory on Tori. In Sec. 3 we discuss some aspects of U-duality for toroidal compactifications in M(atrix) theory. We consider in some detail the seemingly trivial example of M(atrix) theory on T^3 where the condition of geometrising U-duality is analogous to the relation between the IIB string and F-theory. This provides the motivation for the approach pursued in this paper. In Sec. 4, we discuss three orientifolds of type II theory which are dual to the $E_8 \times E_8$ Heterotic string compactified on Tori. We show how these orientifold compactifications of type II theory can be best described as M- or F- theory compactifications on U-manifolds. In Sec. 5, we come to the main result of the paper. Here we show how in Heterotic M(atrix) theory away from its orbifold limit can be described by using the same U-manifolds which occurred in Sec. 4. We also relate to existing proposal for Heterotic M(atrix) theory. In Sec. 6 we end with some concluding remarks. An appendix is devoted to some relevant details of F-theory compactified on an elliptic K3.

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2　Heterotic M(atrix) theory on Tori

We shall collect some of the relevant details of Heterotic M(atrix) theory on tori. See [11, 12] for more details. The Lagrangian for M(atrix) theory on T^d (for $d < 4$) is given by U(N) SYM with 16 unbroken supercharges in $d + 1$ dimensions. This can be obtained from the dimensional reduction of the ten dimensional SYM with $\mathcal{N} = 1$ supersymmetry. The bosonic part of the Lagrangian is given by

$$L_b = \frac{1}{g^2} \int_{\widetilde{T}^d \times R} d^d\sigma dt \ tr \left\{ -\frac{1}{4}F^2 + \frac{1}{2}(D_\alpha X^i)^2 - \frac{1}{4}([X^i, X^j])^2 \right\} \quad , \qquad (2)$$

where X^i for $i = 1, \ldots, (9 - d)$ are adjoint valued scalars and D_α is the covariant derivative with $\alpha = 0, \ldots, d$. The base of the SYM is indicated by $\widetilde{T}^{d}.$[4] When the torus is rectangular, its sides Σ_α are related to that of the target space torus T^d, R_α by

$$\Sigma_\alpha \sim 1/(rR_\alpha) \quad , \qquad (3)$$

in units where $l_{11} = 1$ (and ignoring constants). The eleven dimensional momentum $p_{11} = N/r$ and r is the radius of the eleventh dimension which emerges in the strong coupling limit as argued in ref. [2]. The SYM coupling g is related to the spacetime parameters by

$$g^2 \sim \frac{r^{3-d}}{V} \quad , \qquad (4)$$

where V is the volume of T^d.

The Heterotic string on T^{d-1} is obtained in M-theory as the orbifold $(S^1)/\mathbb{Z}_2 \times T^{d-1}$[24]. The orbifolding is implemented in M(atrix) theory by appropriately embedding the $\mathbb{Z}_2$ in the U(N) SYM in $d+1$ dimensions. (We closely follow the ref. [12] here.) The result of this orbifolding is that Heterotic M(atrix) theory is described by SYM on base $\mathcal{B} = \widetilde{S}^1 \times (\widetilde{T}^{d-1}/\mathbb{Z}_2)$. The bosonic part of the Lagrangian is given by

$$L_b = \frac{1}{g^2} \int_{\mathcal{B} \times R} dt d^d\sigma \ tr \left\{ -\frac{1}{4}F^2 + \frac{1}{2}(D_\alpha X^i)^2 - \frac{1}{4}([X^i, X^j])^2 \right\} \quad , \qquad (5)$$

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[4]We shall indicate the dual tori with a tilde in most cases. Since there are two different but related manifolds in the picture: we shall refer to the one in physical spacetime as the target manifold and the one associated with SYM as the base manifold.

with the sides Σ_α and SYM coupling constant as given before. ($\alpha = 1$ will represent the orbifold direction in target space.) At the $r \equiv 2^{d-1}$ fixed circles of the orbifold group, the U(N) gauge symmetry is broken to $O(N)$ with half of the supersymmetry being broken. The theory as we described it is anomalous. As in any orbifolding with fixed points, one includes a twisted sector localised at the fixed points. The twisted sector is provided by 32 fermions transforming in the fundamental representation of $O(N)$ which seems to be (almost) uniquely determined by anomaly cancellation. (These are equivalent to 16 chiral bosons in $1 + 1$ dimensions after bosonisation.) They are described by the action

$$\sum_{i=1}^{32} \int dt d\sigma^1 \, (\chi_i^I \, (D_+)^{IJ} \, \chi_i^J) \quad , \tag{6}$$

where χ_i are the fermions and D_+ is the covariant derivative. Local anomaly cancellation at each of the fixed circles seems to suggest that the 32 fermions be distributed equally among the r circles (this is only possible for $d < 5$). We will assume that this is the case. Then one obtains a global symmetry $SO(32/r)$ at each of the fixed circles. This implies a gauge symmetry in target spacetime following the rule that global symmetry on the D-branes should be interpreted as gauge symmetries in target space. Hořava has provided a nice description of the situation where the fermions are not localised at the fixed circles using anomaly inflow arguments[12]. A more detailed analysis has been done by Kabat and Rey for the S^1 case where they find a relationship between the gauge coupling and Chern-Simons term (which they introduce) based on *local* anomaly cancellation[14]. In addition, as the fermions are moved away from the fixed points, they pick up masses (for the same reason that open strings connecting separated D-branes pick up a mass.)

3   Manifest U-duality in M(atrix) theory

As mentioned in the introduction the key success of M(atrix) theory is in realising U-duality as a geometric symmetry. For M(atrix) theory compactified on T^2, the U-duality group is $SL(2, \mathbb{Z})$ which is the mapping class group of the base two torus of SYM. M(atrix) theory on T^4 is described by a $(2,0)$ theory in 5+1 dimensions compactified with base $\widetilde{T}^5$. The U-duality group

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$SL(5, \mathbb{Z})$ is the mapping class group of the base torus. For the case of T^5, the U-duality group $SO(5, 5; \mathbb{Z})$ is the T-duality of a six dimensional string theory compactified on a base $\widetilde{T}^5$. Thus these cases all realise U-duality as a geometric symmetry of the underlying field/string theory. It is still an open problem to understand the case of M(atrix) theory on T^d for $d > 5$.

We will now consider the case of $d = 3$ not discussed above. This is an interesting situation which lies at the edge of where the SYM prescription breaks down. The U-duality group here is $SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$ and M(atrix) theory is described by $\mathcal{N} = 4$ SYM in 3+1 dimensions with base $\widetilde{T}^3$. The $SL(3, \mathbb{Z})$ is geometrically realised as the mapping class group of the base torus while the $SL(2, \mathbb{Z})$ corresponds to the electric-magnetic duality which is a quantum non-perturbative symmetry of $\mathcal{N} = 4$ SYM in 3+1 dimensions. However, as is clear, this description has one shortcoming: the full U-duality group is not realised as a geometric symmetry. This is similar to the case of IIB string theory in ten dimensions which has a $SL(2, \mathbb{Z})$ symmetry.

For the IIB string, Vafa proposed F-theory in order to geometrically realise the $SL(2, \mathbb{Z})$[22]. F-theory can be considered to be a IIB compactification where the dilaton and RR scalar are not constant, but vary with respect to the internal manifold. Given a manifold $\mathcal{Y}$ which is elliptically fibred with base $\mathcal{B}$, F-theory on $\mathcal{Y}$ can be defined to be IIB compactified on $\mathcal{B}$ with the dilaton/RR scalar being set equal to the complex structure of the fibre torus, The $SL(2, \mathbb{Z})$ of IIB theory is the geometric symmetry of the torus. For the ten dimensional IIB string this is a simple way to geometrically view the $SL(2, \mathbb{Z})$ as acting on some auxiliary torus. (Of course, it is possible that this theory might exist as a 12 dimensional theory.) A non-trivial example is furnished by considering F-theory on an elliptic K3. This becomes a consistent compactification provided one includes 24 D7 branes and has been shown to be dual to the $E_8 \times E_8$ Heterotic string on a two torus.

Similar considerations can be extended to IIA theory which can be viewed as M-theory compactified on a circle. Non-trivial IIA compactifications can be obtained by considering a fibred manifold $\mathcal{Y}$ with base $\mathcal{B}$ and fibre S^1. As in F-theory, M-theory on $\mathcal{Y}$ can be viewed as IIA on $\mathcal{B}$ with the string coupling related to the radius of the fibre. We shall refer to these manifold on which F- and M- theory are compactified as U-manifolds following ref. [23] since U-duality is manifest as a geometric symmetry of these manifolds.

Inspired by F-theory, we return to the case of M(atrix) theory on T^3,

where we propose to introduce an auxiliary torus whose complex structure is the complexified coupling constant of the SYM theory. This is also motivated by the work of Verlinde who discussed electric-magnetic duality for U(1) SYM in 3+1 dimensions[25]. One begins with a 5+1 dimensional theory with a self-dual two-form gauge field which reduces to the $U(1)$ gauge field and its magnetic dual on compactifying on an auxiliary torus. In the supersymmetric context, the self-dual two-form gauge field becomes part of a tensor multiplet which is the only matter multiplet in $(2,0)$ theories in 5+1 dimensions. For the $U(N)$ case, the appropriate theory is the same $(2,0)$ theory proposed in ref. [9] to describe M(atrix) theory on T^4! Thus requiring manifest $SL(2,\mathbb{Z})$ implies that we should consider the $(2,0)$ theory proposed in ref. [9] compactified on 'T^{2}'$\times \widetilde{T}^3$ where 'T^{2}' is an auxiliary torus on which electric-magnetic duality is geometrically realised and $\widetilde{T}^3$ is the base of SYM.[5]

One might wonder as to why we seem to have considered this result in so much detail. The reason is that just as M-/F-theory provides a large number of non-trivial examples of type II compactifications with lesser supersymmetry, this picture will prove useful in understanding M(atrix) theories with lesser supersymmetry. As it turns out the same U-manifolds which occur in M-theory and F-theory turn up in Heterotic M(atrix) theory!

4 Orientifolds in type II string theory

In this section, we will consider some relevant details of the theories dual to the heterotic string on T^d for $d = 1, 2, 3$. These dual theories have a special point in their moduli space where they can be understood as orientifolds of type II theory. At generic points in the moduli space, (even those corresponding to infinitesimal deformations away from the orientifold limit!) the orientifold cannot be described in such a simple fashion. We will discuss how one can describe these using M-theory and F-theory on certain manifolds which we refer to as U-manifolds following [23] because the U-duality group is realised as a geometric symmetry on these manifolds. We shall follow the notation $\mathcal{I}_n$ to indicate inversion in $n-$directions, Ω refers to the worldsheet parity operation in string theory and F_L is the spacetime fermion number of

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[5]This observation has also been made by E. Verlinde in his seminar at Strings '97.

the left movers in string theory.

4.1   IIA on $S^1/(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_1$

Type IIA with string coupling λ_{10}^A is constructed by compactifying M-theory on a circle of radius R_1. The IIA parameters are given by $\lambda_{10}^A = R_1^{3/2}$ and the ten dimensional Planck length $l_{10}^A = R_1^{-1/2}$ in units where $l_{11} = 1$. Compactifying IIA on a circle of radius R_1^A (in ten dimensional units) is represented by compactifying M-theory on a second circle of radius R_2 (in eleven dimensional units). The two radii are related by $R_1^A = R_2 R_1^{1/2}$. One can obtain the following map which relates operations in IIA theory to those in M-theory by studying the action of the fields in the two theories. (C is the 3-form gauge field of eleven dimensional supergravity)[6]

IIA operation	M-theory operation
$(-)^{F_L}$	$C \to -C$ and $X^1 \to -X^1$
Ω	$X^1 \to -X^1$
$Y^m \to -Y^m$	$X^{m+1} \to -X^{m+1}$

(Note that in M-theory, in order make $\mathcal{I}_n$ (for odd n) a symmetry, one has to take $C \to -C$. In the following this will be assumed even if we do not explicitly indicate it.) We follow the convention that the coordinates of the IIA theory will be represented by Y^m (with radius R_m^A in units where the ten dimensional Planck length $l_{10}^A = 1$, if compactified) and those of M-theory by X^M (with radius R_M, if compactified).

The map discussed above will enable us to construct models dual to IIA compactifications in M-theory. Using the table, one can see that the orientifold of IIA on $S^1/(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_1$ is given by M-theory on $S^1 \times (S^1/\mathcal{I}_1)$ in the limit when $R_1 \ll R_2$. To make this IIA compactification consistent, one has to include 8 D8-branes at each of the two fixed points to cancel RR charge which is localised at the fixed points of the orientifolding group. Each of the fixed points provides a $SO(16)$ gauge group leading to an enhanced gauge symmetry of $SO(16) \times SO(16)$ at the orientifold point.

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[6]Ω is not a symmetry of IIA string theory but $\Omega \cdot \mathcal{I}_1$ is a symmetry. There seem to be two choices for the operation Ω in M-theory: $X^1 \to -X^1$ or $C \to -C$, both of which map $\Omega \cdot \mathcal{I}_1$ to symmetries of M-theory. We believe that the two choices are continuously connected in the moduli space of the U-manifolds and thus are equivalent.

In M-theory, one can consider another limit which is given by $R_1 \gg R_2$, which corresponds to the Heterotic string on S^1 with parameters

$$R^H = R_1 R_2^{1/2} \quad , \tag{7}$$

$$\lambda_{10}^H = R_2^{3/2} \quad , \tag{8}$$

$$l_{10}^H = 1/R_2^{1/2} \quad , \tag{9}$$

where R_H is the radius of the S^1, λ_{10}^H is the ten dimensional heterotic string coupling and l_{10}^H is the ten dimensional Planck length. The nine dimensional string coupling is given by $\lambda_9^H = R_2^{5/4}/R_1^{1/2}$. The relations between the parameters of the Heterotic string and the IIA orientifold that we obtain here are consistent with the relations mentioned in ref. [20]. This is an independent check of the map between IIA operations and M-theory operations which we derived earlier.

As is clear, the above correspondence is at the orientifold limit. One however needs to understand how to describe things away from the orientifold limit. For example, one needs to understand the mechanism of enhancement of gauge symmetry here in IIA language. As we saw, one needed to include 16 D8-branes for cancellation of -8 units of RR charge at the orientifold fixed points. Using S and T-duality, Polchinski and Witten have discussed some aspects of the situation away from the orientifold limit. The important result they obtain is that dilaton picks up a dependence on the coordinate X^2 (in our notation) in regions between the D8-branes after they have moved away from the orientifold plane[20]. Seiberg studied this situation by considering the worldvolume theory of a probe D4-brane and observed that enhanced gauge symmetries which involved the exceptional groups E_n occurred at points where the string coupling diverged[26]. Other groups could be explained by the coincidence of D8 branes either away from the orientifold plane (A_n case) or at the orientifold plane (D_n case). Subsequently it was shown that non-perturbative effects split the orientifold plane into two, releasing an extra 8-brane![27]

All this intricate structure is missing is a geometric picture where the Narain moduli space is visible and duality is manifest. To achieve this we propose that there exists a two dimensional U-manifold $\mathcal{Y}_1$ which has the following properties:[7]

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[7]All D-branes of type IIA string theory except the D8-brane can be understood directly in eleven dimensional supergravity. A related issue is that (at the moment) massive IIA

1. $\mathcal{Y}_1$ is a fibred manifold with fibre S^1 and base $S^1/\mathcal{I}_1$.

2. The moduli space associated with this manifold is

$$\frac{SO(17,1)}{SO(17) \times SO(1)} \times \mathbb{R}^+ \quad .$$

3. There are points in the moduli space where a certain number of 2-cycles (with intersection matrix given by the ADE Cartan matrix) shrink to zero size. (We expect that this is related to the coalescing of degenerate fibres.)

4. There is a point in the moduli space corresponding to constant fibre where at the two end-points of the base, there are shrinking two cycles with the intersection matrix given by the D_8 Cartan matrix.

Assuming that $\mathcal{Y}_1$ exists, then we would like to claim that M-theory compactified on $\mathcal{Y}_1$ should represent the generic situation away from the orientifold limit. We identify points where the fibre degenerates with the location of the D8-branes. Enhanced gauge symmetries correspond to the massless particles which arise when M2-branes wrap around vanishing two-cycles of $\mathcal{Y}_1$.[8] Intuitively, it is easy to see that in the bulk the only possible intersection matrix is of the A_n-type. For example, consider the situation where two degenerate fibres coalesce on the base away from the end-points. This corresponds to the A_1 case where a single two-cycle shrinks to zero. This argument shows that D_n and E_n situations cannot arise in the bulk. This seems to agree with the discussion in ref. [26]. However, beyond this qualitative picture we do not have a quantitive correspondence with the details presented in ref. [20].

Let us summarise the properties of the function which describes the radius of the fibre circle on $\mathcal{Y}_1$. We caution the reader that this has not been derived but guessed by using our picture of symmetry enhancement and the analysis of refs. [20, 26, 27].

supergravity cannot be derived from eleven dimensional supergravity. These issues need to be understood in order to understand quantitative features of $\mathcal{Y}_1$[28].

[8]As this manuscript was being prepared, a paper by Ashoke Sen has appeared which discusses a similar mechanism for enhancement of gauge symmetry for the orientifold $T^3/(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_3$[29].

1. It possesses 18 zeros corresponding to the positions of the 16 perturbative 8-branes and the 2 non-perturbative ones.

2. It is a continuous function with cusps at each of the zeros with the discontinuity in the first derivative being related to the 8-brane charge.

Is there a candidate for $\mathcal{Y}_1$? Interestingly, there is one which was discussed by Morrison and Vafa[30]. We have not checked that it satisfies all our conditions and will leave it for the future since this is not the main focus of this paper. The candidate is a real version of elliptic K3 discussed in the appendix. Consider the manifold

$$y^2 = x^3 + f(z) \, x + g(z) \quad . \tag{10}$$

where $f(z)$ and $g(z)$ are homogeneous polynomials (in z) of degrees eight and twelve respectively. Here x, y, z are considered to be real numbers (unlike in the appendix where they are complex). The equation is considered to be the equation of a circle with z as the coordinate on the base. The number of parameters which describe this is 18 (22 parameters fix f and g and one has to subtract three to account for a global $SL(2, \mathbb{R})$ acting on z and one for an overall rescaling.). So this passes the simple test that it gets the dimension of the moduli space correctly. Let us assume that Tate's algorithm can be applied in this case. Since in the orientifold limit of $\mathcal{Y}_1$, we expect an $SO(16)$ singularity at the ends, Tate's algorithm implies that the discriminant has zeros of order ten at the two end-points of the base. The ten zeros can be identified with the nine 8-branes and the orientifold 8-plane. Again, this seems consistent with picture we mentioned earlier.

4.2 IIB on $T^2/(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$

Following Aspinwall and Schwarz[31], we construct IIB on S^1 in M-theory as follows. Consider M-theory compactified on a two-torus with sides R_1 and R_2 respectively. In the limit $R_1 \ll R_2$ and $R_1, R_2 \to 0$, one obtains type IIB on S^1 with length $1/(R_1 R_2)$ in units where $l_{11} = 1$ ($l_{10}^B = 1/R_1^{\frac{1}{2}}$). Converting to units where $l_{10}^B = 1$, we get the radius of the S^1 to be $R_1^B = 1/(R_1^{\frac{1}{2}} R_2)$.

The symmetries of IIB theory on S^1 can now be seen as symmetries of M-theory on T^2. The map obtained by studying the action on the fields of eleven dimensional supergravity is

IIB operation	M-theory operation
$(-)^{F_L}$	$C \to -C$ and $X^1 \to -X^1$
Ω	$C \to -C$ and $X^2 \to -X^2$
$SL(2, \mathbb{Z})$	Mapping class group of the M-theory torus.
$Y^1 \to -Y^1$	$C \to -C$, $X^2 \to -X^2$ and $X^1 \to -X^1$.
$Y^m \to -Y^m$, $m \neq 1$	$X^{m+1} \to -X^{m+1}$

We follow the convention that the coordinates of the IIB theory will be represented by Y^m (with radius R_m^B in units where the ten dimensional Planck length $l_{10}^B = 1$, if compactified) and those of M-theory by X^M (with radius R_M, if compactified). This will enable us to construct models dual to IIB compactifications in M-theory. For example, IIB compactified on an orbifold K3= $T^4/\mathcal{I}_4$ can be realised in M-theory as a compactification on $T^5/\mathcal{I}_5$ which is in agreement with known dualities[32].

Thus, type IIB compactified on a two-torus of sides R_1^B and R_2^B (in units where $l_{10}^B = 1$) can be obtained from M-theory compactified on a three torus with sides R_1, R_2 and R_3 (in units where $l_{11} = 1$). The parameters are related as follows

$$
\begin{aligned}
R_1^B &= 1/(R_1^{1/2} R_2) \\
R_2^B &= R_1^{1/2} R_3 \\
\tau_2 &\equiv \mathrm{Im}(\tau) = R_2/R_1
\end{aligned}
\tag{11}
$$

Note that we have chosen a simple situation where all the tori are rectangular.

We are interested in constructing the orientifold: type IIB on $T^2/(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$. Using the map relating operations in IIB to those in M-theory given above, we see that $(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$ maps to an inversion of X^8 (i.e., the coordinate associated with the radius R_3 in the M-theory compactification). Thus, we see that the orientifold is obtained from M-theory compactified on $T^2 \times (S^1/\mathbb{Z}_2)$ in the limit where the T^2 has vanishing area.

This theory has another limit given by $R_3 \to 0$ which corresponds to the $E_8 \times E_8$ heterotic string on T^2 with Wilson lines breaking the gauge group to $SO(8)^4$. In this limit, the ten-dimensional heterotic coupling $\lambda_{10}^H = R_3^{3/2}$ and the ten-dimensional Planck length $l_{10}^H = 1/R_3^{1/2}$. Now, one can relate the parameters of the two theories using M-theory. We obtain

$$
V_{T2}^B \, \tau_2^{1/2} = \lambda_8^H \quad , \tag{12}
$$

where $V_{T2}^B = R_1^B R_2^B$ is the area of the IIB torus. Further, the complex structure of the heterotic torus is clearly the modular parameter of the IIB theory. The complex structure of the IIB torus

$$\rho_2 \equiv \text{Im}(\rho) = R_2^B/R_1^B = R_1 R_2 R_3 = V_{T2}^H \quad ,$$

where V_{T2}^H is the volume of the heterotic torus (in units where $l_{10}^H = 1$). Thus, the complexified Kähler structure of the heterotic string theory is mapped to the complex structure of the IIB torus. All the relations we have derived are consistent with the results of Sen mentioned in the appendix.

As in the type IIA situation, one would like to be able to describe the situations away from the orientifold limit. The U-manifold $\mathcal{Y}_2$ for this was proposed by Vafa as the first example of an F-theory compactification[22]. $\mathcal{Y}_2$ here is an elliptic K3 with enhancement of gauge symmetry corresponding to ADE singularities on the U-manifold. The important issue is that from the IIB viewpoint, there are regions which involve strong coupling as was shown in a beautiful paper by Ashoke Sen[21]. Subsequently, there was a nice reinterpretation of Sen's results using the worldvolume theory of a D3-brane probe[33]. An important result from this analysis is as follows: At the orientifold point, one has four D_4 singularities which can be understood as coming from four coincident D7 branes at the orientifold plane. Non-perturbative effects (from the IIB viewpoint) split the orientifold plane into two (just as in the IIA case discussed earlier)! Enhanced gauge symmetry occurs when two-cycles of $\mathcal{Y}_2$ shrink to zero size with the gauge group given by the intersection matrix. Some relevant details are discussed in the appendix.

4.3   IIA on $T^3/(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_3$

We include the case of IIA on $T^3/(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_3$ to include another example in our list. This orientifold can be mapped to M-theory on $S^1 \times (T^3/\mathcal{I}_3)$. The enhanced gauge symmetry here is $SO(4)^8 = SU(2)^{16}$. One can again ask what happens when one goes away from the orientifold limit. Equivalently, we can ask if we can find a four dimensional fibred U-manifold $\mathcal{Y}_3$ with fibre S^1 which describes the situation of varying IIA coupling. The answer here is that $\mathcal{Y}_3$ is a K3. One valid objection to this is that a generic K3 is not a fibred manifold with fibre S^1. However, it is possible that in the complete moduli space of K3, there is a subspace (of codimension zero) which satisfies

this criterion. There is a recent paper by Aspinwall which might be relevant in understanding this[34]. Further, this is consistent with the known duality of the Heterotic string on T^3 and M-theory on $K3$.

4.4   A summary

For the reader who is not interested in the technical details, we now present a summary of the results which we have discussed in the previous subsections. We provided maps which directly relate orientifolds of type II string theory to the $E_8 \times E_8$ heterotic string at special points in their moduli spaces using M-theory. These maps can also be obtained by a complex sequence of T and S duality operations involving the type I and the two Heterotic strings. (An example due to Sen is provided in the appendix. A similar sequence has also been discussed by Polchinski and Witten[20].) Away from the orientifold limits, these orientifolds are best understood as compactifications of M- or F- theory on U-manifolds $\mathcal{Y}_d$. One common feature which emerges in the examples considered is that enhancement of gauge symmetry can be understood in these theories as coming from the wrapping of the M2-brane on shrinking two-cycles of the U-manifold. From the type II picture this is understood as coming from the coinciding of D-branes. As we will see later, in M(atrix) theory the first viewpoint is precisely the mechanism by which enhanced gauge symmetry and (not surprisingly) the twisted sector fermions emerge.

5   Implications for M(atrix) theory

In the previous section, using U-manifolds, we saw how one could describe situations corresponding to deformations away from orientifold limits inspite of the fact that the string theory could be at strong coupling. In M-theory we achieved this by considering a fibred manifold with fibre an S^1 whose radius was related to the string coupling constant of IIA theory. In F-theory, this was achieved by considering another fibred manifold with fibre T^2 with the complex structure of the fibre torus being the dilaton-RR scalar of IIB string theory. Enhanced gauge symmetry is related to the vanishing of certain cycles on the U-manifold and the moduli space is recovered as the moduli space of the U-manifold.

In this section, we will discuss the implications of these observations for M(atrix) theory. We discuss the case of Heterotic M(atrix) theory on T^2 and then discuss the cases of S^1 and T^3. The important result which we will achieve is that U-duality will be manifest and further the twisted sector ("fermions") appear naturally without being postulated.

The basic idea is that in Heterotic M(atrix) theory, moving away from the orbifold limit corresponds to letting the coupling constant of the SYM vary with the base such that the appropriate amount of supersymmetry is preserved. Additional constraints come from anomaly cancellation. (Issues related to this have been discussed by Hořava[12] and by Kabat and Rey[14].) Further, there can be regions where the coupling constant might be strong. This might seem to be a rather difficult problem to solve. Actually, this is rather similar to what we observed for the case of the type II orientifolds we considered in Sec. 4. So, as in that case, we propose to consider an auxiliary manifold whose fibre is the coupling constant of the SYM. For Heterotic string theory on T^d the auxiliary manifold $\mathcal{Y}_d$ has to satisfy the following conditions:

1. $\mathcal{Y}_d$ is a fibred manifold with fibre S^1 (for odd d) and T^2 (for even d). (This is related to the fact that in four dimensions one can complexify the coupling constant of SYM by including the θ - term $F \wedge F$.)

2. The moduli space associated with this manifold is

$$\frac{SO(16+d, d)}{SO(16+d) \times SO(d)} \times \mathbb{R}^+ \quad .$$

3. There are points in the moduli space where certain number of 2-cycles (with intersection matrix given by the ADE Cartan matrix) shrink to zero size.

4. There is a point in the moduli space corresponding to constant fibre where the base of the manifold is $T^d/\mathcal{I}_d$ with two cycles shrinking to zero size at the fixed points whose intersection matrix is given by the $SO(32/r)$ Cartan matrix where $r = 2^d$.

5. Enhanced gauged symmetry should occur at precisely the same point where the dual type II compactification has gauge symmetry enhancement.

Following the discussion in Sec. 4, the U-manifolds we described for the type II orientifolds seem the perfect candidates. However, one has to decide on which theory these manifolds have to be compactified. The answer is fixed by the condition that for large base (small fibre) and away from possible singular fibres, the theory should look like SYM on T^d (for $d < 4$). A natural candidate which satisfies this is the theory which describes M(atrix) theory on T^s for some $s > d$. This leads us to propose the following as a replacement of the SYM prescription.

Compactify M(atrix) theory on $S^1 \times \mathcal{Y}_d$,

where the theory to be chosen is decided using the criterion mentioned above. This can be considered as the *main result* of this paper. For example, for $\mathcal{N} = 8$ SYM in 2+1 dimensions is replaced by $\mathcal{N} = 4$ U(N) SYM in 3+1 dimensions (this theory describes M(atrix) theory on T^3).

Immediate consequences are: U-duality is manifest since it is encoded in the geometry of $\mathcal{Y}_d$. Unlike in the case of the orientifolds, we however seem to have an extra modulus associated with the size of the S^1. The resolution is that in all the examples that we consider, the theory is superconformal and hence one of the scales is not a modulus. The proposed theory is non-anomalous since we expect compactification to preserve the non-anomalousness of M(atrix) theory on T^s. Thus, these are simple consistency checks which our proposal passes. We shall now consider the proposal for three cases $d = 1, 2, 3$. We first discuss the case of $d = 2$ since many properties and issues can be made explicit.

5.1   Heterotic M(atrix) theory on T^2

In M(atrix) theory, the heterotic string on T^2 is described by the orbifold SYM theory on $\widetilde{S}^1 \times \widetilde{T}^2/\mathbb{Z}_2$. Anomaly considerations requires one to include 32 fermions in the theory as a twisted sector. Distributing the fermions equally among the four fixed circles leads to an $SO(8)^4$ gauge symmetry in the target space. This is precisely the same gauge group which arose in the IIB orientifold we considered and was the motivation for the special limit for F-theory considered in the appendix.

From Sec. 4, we see that $\mathcal{Y}_2$ is an elliptic K3 and we should consider the M(atrix) theory describing M-theory on T^4. This is a theory with (2,0)

supersymmetry in 5+1 dimensions which on compactifying on a two torus reduces to SYM as mentioned earlier[10]. So our proposal implies that Heterotic M(atrix) theory on T^2 is described by *compactifying the (2,0) theory on $S^1 \times \mathcal{Y}_2$*. It clearly satisfies all the conditions we described. Among them it satisfies the constraint that it possess a limit with constant fibre such that one has $SO(8)^4$ enhanced gauge symmetry with base $T^2/\mathbb{Z}_2$. As shown in the appendix this limit exists. Berkooz and Rozali have also considered compactifying the (2,0) theory on an elliptic K3 in relation to F-theory on K3[16]. However, they attempted to relate to the heterotic string at the point where the enhanced gauge symmetry is $E_8 \times E_8$. Thus it was not possible to relate to the existing prescriptions which lead to $SO(8)^4$. Our proposal is however consistent with their result.

Another important consequence is that the twisted sector fermions (which are equivalent to chiral bosons in 1+1 dimensions) can be seen clearly. At the orientifold limit, one has four two-cycles shrinking to zero with their intersection matrix given by the D_4 Cartan matrix. For the U(1) case, the (2,0) theory is a theory of a single tensor multiplet which has a self-dual two-form gauge field. Away from the singularities, this provides SYM as described by Verlinde[25]. At the singularities, the same two-form gauge field provides four chiral bosons which on fermionising transform in the 8 of D_4. We thus recover the twisted sector fermions which had to be postulated at the oribifold limit. The case for U(N) needs a better understanding of the (2,0) theory but it is not unlikely that the bosons will transform in the appropriate representation.

Let us now attempt to return to the SYM picture. When the base of $\mathcal{Y}_2$ (which is a $\mathbb{CP}^1$) is large, the geometry of $\mathcal{Y}_2$ dictates the variation of the SYM coupling constant $\rho = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. An important condition is that ρ varies holomorphically with respect to the complex coordinate of the base $\mathbb{CP}^1$. Thus contrary to the naive guess that the base manifold of the SYM is $\widetilde{T}^2/\mathcal{I}_2 \times S^1$, it is given by $\mathbb{CP}^1 \times S^1$. At 24 points (corresponding to the zeros of the discriminant of the elliptic K3), the SYM is at strong coupling. Enhanced gauge symmetries correspond to the coalescing of zeros with the gauge group given by Tate's algorithm[30]. In the SYM picture, this corresponds to a global symmetry acting on chiral bosons (fermions) living on the S^1 located at the zero of the discriminant. For example, an E_8 symmetry occurs when ten zeros of the discriminant coincide with the order of the zeros of f is larger than three and that of g is equal to five at the same location. Thus in the

moduli space, there is a point where $E_8 \times E_8$ occurs[30]. We interpret the condition that ρ depends holomorphically on the coordinate of the $\mathbb{CP}^1$ as a BPS condition which breaks half of the $\mathcal{N} = 4$ supersymmetry. It is of interest to derive this from first principles in 3+1 SYM. We hope to report on some of these aspects in the future.

5.2 Heterotic M(atrix) theory on S^1

Using an approach different from ours, Kabat and Rey have considered this problem directly from the SYM in 2+1 dimensions. We thus will be able to compare our proposal to their results. Here $\mathcal{Y}_1$ is the two dimensional U-manifold which we discussed in Sec. 4. Our proposal suggests that we compactify 3+1 dimensional U(N) SYM on $S^1 \times \mathcal{Y}_1$. When the base of $\mathcal{Y}_1$ is large and away from singular fibres, locally the theory reduces to SYM in 2+1 dimensions using standard dimensional reduction. Further, at the limit where $\mathcal{Y}_1$ has constant fibre, at the endpoints of the base, one has eight 2-cycles whose intersection matrix is given by the D_8 Dynkin diagram, the field strength of the gauge field can provide scalars living on the 1+1 dimensional circle. We do not understand this mechanism very well and thus this should only viewed as a possible scenario. We do not see how the chirality of the bosons emerge.

We now comment on the relationship to the recent work of Kabat and Rey[14]. Since our proposal leads to a varying coupling constant it is similar to their work. However, since we do not fully understand $\mathcal{Y}_1$ as yet, we cannot make a precise comparision. Kabat and Rey point out that T-duality is realised as S-duality in Heterotic M(atrix) theory. This can probably be understood in our proposal as a part of the electric-magnetic duality of SYM in 3+1 dimensions thus providing a different perspective to their result. (This might also be related to the results of Susskind and Ganor et al.[6], who showed that T-duality in M(atrix) theory on T^3 is realised as S-duality of 3+1 dimensional SYM.)

Kabat and Rey obtain a potential for the scalars and also find different behaviour for the centre of mass U(1) and the SU(N) coupling constants. We do not understand how this feature will emerge in our situation. This might need a better understanding of the of $\mathcal{Y}_1$ and details of how the dimensional reduction works. However, since in the analysis of Kabat and Rey, this follows from rather general considerations, we expect that such behaviour

should emerge.

The only disagreement we have with their result is the choice of the variation for the coupling constant of the SYM (which is related to z(y) in their notation). Even though we cannot make a precise statement about the functional form of this function since we do not know the detailed structure of $\mathcal{Y}_1$, our mechanism of gauge symmetry enhancement provides insight into the form of the variation of coupling constant. However, the crucial features which are required in the variation for the coupling constant as follows from the analysis on Kabat and Rey seem to be present. Thus we do not consider it to be a major disagreement with their analysis.

We will now discuss the feature of the variation of the coupling constant in our proposal. In our proposal, the radius of the fibre is related to the coupling constant of SYM (and string coupling in the IIA orientifold). At a zero, SYM is a strong coupling since the 2+1 dimensional coupling constant is related to the inverse of the radius ($g^2 \sim 1/R$, assuming standard dimensional reduction). Consider the A_1 situation which occurs when two degenerate fibres coalesce. Since the radius of the fibre is positive definite, it will have to increase away from the zero but at some point it should turn around to touch zero again. This is the picture forced on us by the occurance of two neighbouring zeros. We do not see this behaviour in the variation of the coupling as chosen by Kabat and Rey. However, there must be a neighbourhood of near a zero where one can approximate the function locally by a linear function such that locally it agrees with the form of Kabat and Rey. In the analysis of Kabat and Rey, the jump in the derivative was related to the Chern-Simons coefficient and thus this feature is captured in our scenario. A more detailed comparision will require understanding $\mathcal{Y}_1$ as well as the issues related to D8-branes and massive IIA supergravity being obtained from eleven dimensional supergravity[28].

5.3   Heterotic M(atrix) theory on T^3

Here $\mathcal{Y}_3$ is a K3 and hence our proposal would correspond to considering the (2,0) theory on $S^1 \times K3$ which is in agreement with an existing proposal[15, 16, 17].[9] From the arguments given in sec. 4, Heterotic theory will be visible

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[9]This example is rather different from the first two examples we just described. This theory is not expected to be described by SYM even in its orbifold limit. Thus what we

only in the subspace in the moduli space of K3 where it is a fibred manifold with fibre S^1. This is in agreement with related remarks of Berkooz and Rozali[16]. The mechanism of gauge symmetry enhancement here is similar to the case of Heterotic M(atrix) theory on T^2.

6 Conclusion

In this paper, we have discussed Heterotic M(atrix) theory on T^d at generic points in their moduli spaces. We have proposed certain U-manifolds $\mathcal{Y}_d$ which have the property that M(atrix) theory compactified on $S^1 \times \mathcal{Y}_d$ describes Heterotic M(atrix) theory on T^d at generic points in its moduli space. Evidence for our proposal is provided by manifest U-duality, a limit where we recover the orbifold SYM on $S^1 \times T^d/\mathbb{Z}_2$ with the correct gauge symmetry and a natural mechanism for the appearance of the postulated fermions precisely at the point where enhanced gauge symmetry occurs (we do not understand this too well for the case of $d = 1$). In conjunction with the results of Kabat and Rey, we hope that this is a step towards understanding Heterotic M(atrix) theory in more detail. One aspect which is lacking here is some kind of first principles derivation of (atleast) some of the features starting from the orbifold SYM. This is being currently studied[35].

One striking feature which seems to come out of this proposal is the disappearance of the spacetime picture. The T^2 case is rather striking since the dual torus gets replaced by a sphere. This is similar to what happens in M(atrix) theory on T^4 and T^5 and must be a generic situation when U-duality is manifest[9, 10]. Spacetime can be only recovered in some special limits as emphasised by these authors.

The proposal of using U-manifolds is more general than the examples we considered in this paper. It might prove useful in situations with even fewer unbroken symmetries. An interesting example is that of the Heterotic string compactified on K3. Now one has to specify additional data corresponding to a choice of vector bundle on K3. This is dual to F-theory on certain elliptic CY 3-folds[30]. A natural guess for describing these cases in M(atrix) theory is to consider M(atrix) theory on the same CY$\times S^1$. However this is a seven dimensional base space and hence we need to understand M(atrix)

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are doing corresponds to an extrapolation of the previous examples. The only justification which we can provide is that the result seems consistent.

theory on T^6 and maybe even T^7. M(atrix) theory in this case might provide another window into understanding the relation between vector bundles on K3 which occur on the heterotic side and the CY 3-folds which occur in the F-theory description. We also believe that our results might be relevant to the understanding of D-branes in curved space[36].

Appendix

F-theory on elliptic K3

We collect some facts about F-theory on an elliptic K3 which are relevant. An elliptic K3 surface is given by the equation

$$y^2 = x^3 + f(z) \, x + g(z) \quad . \tag{13}$$

where $f(z)$ and $g(z)$ are homogeneous polynomials (in z) of degrees eight and twelve respectively. The K3 is the fibred manifold with base $\mathbb{CP}^1$ (and coordinate z) and the fibre is a torus described by the equation given above. The modular parameter of the torus is implicitly given by

$$j(\tau(z)) = \frac{4(24f)^3}{27g^2 + 4f^3} \quad , \tag{14}$$

where j is the standard j-function. Generically, the fibre degenerates at the 24 zeros of the discriminant $\Delta \equiv 4f^3 + 27g^2$.

Since we will be interested in understanding the relationship of F-theory compactified on elliptic K3 to heterotic strings compactified on T^2 in M(atrix) theory, it is of interest to consider a special limit. This is a limit where the modular parameter $\tau(z)$ is constant over the base. Several possibilities exist[21, 37, 30]. We shall however consider the one considered by Sen[21].

This corresponds to the case where the discriminant has four zeros, each of order six.[10]

Explicitly, this corresponds to choosing

$$g = \phi^3 \text{ and } f = \alpha \ \phi^2 \quad , \tag{15}$$

where $\phi = \prod_{i=1}^{4}(z - z_i)$. This leads to an enhanced gauge symmetry of $SO(8)^4$ in the F-theory compactification. In addition, the base can now be considered to be a torus whose complex modulus 'ρ' corresponds to the cross-ratio of the locations of the four zeros of the discriminant. The moduli of this theory are now given by τ, ρ and the size of the base V_{T2}^B. At this special limit, F-theory on K3 can be identified with type IIB string on $T^2/(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$.

We shall now use a chain of dualities to map this F-theory compactification to a compactification of the $E_8 \times E_8$ Heterotic string on a two-torus. This will enable us to obtain an explicit relationship of the moduli on both sides. First, we shall T-dualise on both circles of the base. This maps the $\mathbb{Z}_2$ transformation $(-)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$ to the transformation Ω. Thus we can map the F-theory compactification to type I on the dual torus. We can now use the type I - Heterotic duality to obtain a map to the $SO(32)$ Heterotic string. Finally, by T-dualising on one of the circles, we obtain a map from the F-theory compactification to the $E_8 \times E_8$ Heterotic string. The moduli on the heterotic side given by the complex and Kähler moduli of the torus (on which the heterotic string is compactified) get mapped to the IIB moduli τ and ρ respectively. Further, the eight-dimensional heterotic string coupling λ_8^H is related to the IIB moduli as

$$\lambda_8^H = V_{T2}^B \ \tau_2^{\frac{1}{2}} \quad . \tag{16}$$

The above chain of dualities as well as relations just mentioned have been described by Sen[21].

References

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[10]The fact that Δ has zeros of order six at each z_i is important. Deforming away from the orientifold limit, which in the elliptic K3 picture corresponds to separating the zeros, one obtains six separate zeros. Four of them can be identified with the positions of the D7-branes. Sen has argued that the other two zeros corresponds to a splitting of the orientifold plane or equivalently the emitting of a 7-brane by the orientifold plane.

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